Reduced Dimensional Chebyshev-Polynomial Chaos Approach for Fast Mixed Epistemic-Aleatory Uncertainty Quantification of Transmission Line Networks

Aditi Krishna Prasad, *Student Member, IEEE*, and Sourajeet Roy, *Member, IEEE*

***Abstract*— This paper presents a hybrid Chebyshev-polynomial chaos (CPC) metamodel for the uncertainty quantification of multiconductor transmission line networks. The key feature of this metamodel is its capacity to account for two different types of uncertainty simultaneously present in a network—uncertainty arising from imprecise knowledge regarding the network parameters (epistemic uncertainty) and uncertainty arising from the random variability in the network parameters (aleatory uncertainty). Unfortunately, when investigating such mixed epistemic-aleatory problems, the CPU costs to construct the hybrid CPC metamodel becomes exorbitantly large. To address this issue, a new sensitivity sweeping-based dimension reduction algorithm especially tailored for mixed epistemic-aleatory problems has been developed in this paper. Numerical techniques based on iterative model enrichment and priority-based sampling have also been developed to further enhance the efficiency of the algorithm.**

***Index Terms*— Aleatory uncertainty, dimension reduction, epistemic uncertainty, multiconductor transmission lines (MTLs), polynomial chaos, sensitivity analysis.**

I. Introduction

With the sustained miniaturization of transistor feature sizes below the 22-nm level, uncertainty in interconnect networks has been found to introduce widespread unpredictability in the performance of integrated circuits [1]–[4]. In order to accommodate this uncertainty in the early design cycles, new circuit simulators capable of modeling the impact of uncertainty in multiconductor transmission line (MTL) networks need to be developed. An important aspect of such simulators must be their ability to handle both aleatory and epistemic forms of uncertainty. Aleatory uncertainty refers to the natural variability inherent in the MTL network parameters [5]–[7]. Typically, this type of uncertainty arises from the manufacturing and fabrication process variations and is commonly modeled using random variables with known probability density functions (PDFs), e.g., uncertainty in the area of cross section and the length of conductors in the MTL network [5]–[7]. On the other hand, epistemic uncertainty refers to the lack of data or knowledge regarding the MTL network parameters [5]–[7]. This type of uncertainty occurs due to unavoidable errors in the numerical simulations and measurements when quantifying the electrical and mechanical properties of materials. The epistemic parameters are represented as intervals of possible values with no knowledge of the PDF available. One common example is the uncertainty in the value of the relative permittivity of the dielectric medium used in MTL networks [5]–[7].

Currently, the literature on uncertainty quantification in MTL networks has largely focused on problems exhibiting only aleatory uncertainty [1]–[3], [8]–[19]. Very little attention has given to the more general problems where both epistemic and aleatory uncertainty is simultaneously present within the same network (i.e., mixed uncertainty problems). A characteristic feature of mixed uncertainty problems is that the statistics of the network responses are no longer unique. Rather, these statistics vary with the values of the epistemic parameters. Thus, the main goal of mixed uncertainty quantification (MUQ) is to precisely determine the maximum and minimum bounds circumscribing the variability in any response statistics [5]–[7].

The most common approach for performing MUQ is by using metamodels such as the augmented PC metamodel [20], [21] and the hybrid Chebyshev-polynomial chaos (CPC) metamodel [22], [23]. Of these, the CPC metamodel is particularly attractive due to its theoretically correct treatment of epistemic variables as nonprobabilistic variables [22], [23]. The CPC metamodel represents the effect of the aleatory and epistemic dimensions on any model response using polynomial chaos (PC) and Chebyshev basis functions, respectively. The coefficients of the basis functions form the new unknowns of the system and are usually evaluated using nonintrusive methods [22], [23]. Once the coefficients are determined, the CPC metamodel serves as a closed-form proxy model that can be directly probed for MUQ.

Unfortunately, the main limitation of the CPC metamodel is that the number of unknown coefficients grows very rapidly with the number of uncertain dimensions in a net-

Manuscript received May 16, 2018; revised September 10, 2018 and October 23, 2018; accepted October 28, 2018. Date of publication November 12, 2018; date of current version June 6, 2019. Recommended for publication by Associate Editor S. Grivet-Talocia upon evaluation of reviewers' comments. *(Corresponding author: Sourajeet Roy.)*

A. K. Prasad was with the Electrical and Computer Engineering Department, Colorado State University, Fort Collins, CO 80523-1373 USA. She is now with Intel, Fort Collins, CO 80526 USA.

S. Roy is with the Electrical and Computer Engineering Department, Colorado State University, Fort Collins, CO 80523-1373 USA (e-mail: sjeetroy@engr.colostate.edu).

Color versions of one or more of the figures in this paper are available online at http://ieeexplore.ieee.org.

Digital Object Identifier 10.1109/TCPMT.2018.2880995

work. This means that even for a modest set of uncertain network dimensions, an exorbitant number of deterministic SPICE simulations are required to evaluate all the coefficients [24]–[26]. This issue is further compounded by the fact that conventional techniques to reduce the size of metamodels, developed for purely aleatory problems, cannot be extended to mixed uncertainty problems [16], [18], [27]–[35]. This is because all the conventional methods invariably rely on the statistics of the network response to determine the best possible strategy to reduce the size of the metamodel. Unfortunately, unlike purely aleatory problems, the response statistics of mixed uncertainty problems vary with the local values of the epistemic dimensions. Because of this local variability in the response statistics, so far it has been impossible to extract meaningful and unbiased information to guide a metamodel reduction process. This inability to reduce the CPC metamodel makes MUQ a far more computationally expensive problem than simply aleatory UQ [1]–[3], [8]–[19].

To curb the computational expense involved in MUQ, in this paper, a dimension reduction strategy especially tailored for mixed uncertainty problems has been developed. As a first step of this strategy, a multistage iterative methodology is formulated to construct a crude forward model of the network response. The number of iterations in this methodology is adaptively tuned to find a suitable tradeoff between the accuracy of the forward model and the CPU costs expended in constructing this forward model. Once the forward model is constructed, it is then used to calculate the sensitivity of the response variance with respect to each epistemic and aleatory dimension of the network at many different points distributed over the entire multidimensional epistemic space. This sensitivity analysis algorithm is hereafter referred to as a sensitivity sweeping algorithm. The sensitivity sweeping algorithm tracks how the sensitivity measure (or index) of each dimension varies across the entire multidimensional epistemic space. Based on this global sensitivity information, those epistemic and aleatory dimensions that have negligible impact on the response variance across the entire epistemic space are identified. Thus, the key innovation of the proposed sensitivity sweeping algorithm is that it can correctly identify the unimportant network dimensions even in the presence of locally variable response statistics. Once these unimportant dimensions are identified and removed, a more compact CPC metamodel of the network response can be constructed at a fraction of the expected CPU costs without much loss in accuracy.

While this paper is based on the author's preliminary work of [26], it advances new mathematical ideas, algorithms, and analyses that are not to be found in [26]. First, the sensitivity sweeping algorithm of this paper demonstrates how to identify the unimportant epistemic dimensions in a mixed uncertainty problem–knowledge that was not included in [26]. As a result, the proposed approach is able to reduce both the epistemic and aleatory network dimensions instead of simply reducing the aleatory dimensions as shown in [26]. This translates to far greater reduction in CPU costs for the proposed approach over that of [26] as demonstrated in Section IV. Second, the proposed approach is flexible enough to consider complicated problems where epistemic uncertainty is embedded in both the mean and standard deviation (SD) quantities of aleatory dimensions. This is in contrast to the work of [26] where no epistemic uncertainty was considered in the SD quantity. Third, when considering epistemic uncertainty in both the mean and SD quantities, the 1-D forward model proposed in [26] is grossly inaccurate for sensitivity analysis. Hence, a robust and flexible forward model capable of incorporating 2-D (and possibly even higher dimensional) terms depending on the network is developed in this paper. Finally, when constructing this robust forward model, an iterative enrichment strategy and a priority-based sampling strategy to minimize the CPU costs is used. Note that no such cost-mitigating techniques have been developed for the forward model of [26]. The accuracy and efficiency of this paper is validated via multiple numerical examples in Section IV.

II. Preliminaries Regarding MUQ

Consider a general distributed deterministic MTL network the electrical behavior of which is represented in SPICE using the standard modified nodal analysis equations

$$\mathbf{G}\mathbf{X}(t)+\mathbf{C}\frac{d\mathbf{X}(t)}{dt}+\mathbf{F}(\mathbf{X}(t)) + \sum_{i=1}^{N_t}\left(\mathbf{T}_i\mathbf{Y}_i(t)\mathbf{T}_i^T\right)*\mathbf{X}(t)=\mathbf{B}(t). \quad (1)$$

In (1), $\mathbf{G}$ and $\mathbf{C}$ are the coefficient matrices, $\mathbf{X}$ is the vector of uncertain voltage/current responses, $\mathbf{F}$ contains the stamp of nonlinear circuit elements, $\mathbf{T}_i$ is the selector matrix mapping the vector of port currents $\mathbf{i}_i(t)$ for the ith distributed MTL network into the nodal space of the circuit, $\mathbf{Y}_i$ is the corresponding time-domain $\boldsymbol{Y}$-parameter macromodel of the ith MTL network, $\mathbf{B}$ represents the vector of independent voltage and current sources, and "$*$" denotes the temporal convolution which is performed in a recursive manner in SPICE. Next, it is assumed that epistemic and aleatory uncertainty is introduced into the network of (1) via the geometrical, electrical, material, and device parameters. The mathematical representation of the epistemic and aleatory uncertainty is discussed next.

A. Epistemic and Aleatory Uncertainty Is Separated

Let M and N parameters of the network of (1) exhibit epistemic and aleatory uncertainty, respectively. If all these parameters are different from each other, then the epistemic and aleatory uncertainty in these parameters are said to be separated. In that case, these parameters can be mathematically represented as

$$\begin{aligned} e_{\min,i} \le e_i \le e_{\max,i}\,; \quad 1\le i\le M \\ a_j = a_{j,0}(1+s_j\lambda_j)\,; \quad 1\le j\le N \end{aligned} \quad (2)$$

where $[e_{\min,i}, e_{\max,i}]$ represent the closed interval of values the epistemic parameter e_i can assume, $a_{j,0}$ is the nominal (mean) value of the aleatory parameter a_j, s_j is the corresponding relative SD, and λ_j is the corresponding aleatory dimension. Using a linear transformation, the variability in the epistemic parameter e_i can be mapped to the variability in an epistemic dimension ξ_i as

$$e_i = \frac{(e_{\max,i}+e_{\min,i})}{2}+\frac{(e_{\max,i}-e_{\min,i})}{2}\xi_i \quad (3)$$

where $\xi_i \in [-1, 1]$. Note that there is no available information regarding the marginal PDF of ξ_i while the marginal PDF of λ_j is assumed to be known.

B. Epistemic and Aleatory Uncertainty Coexists in the Same Model Parameter

Besides the M and N parameters of above, let there be an additional R network parameters of (1) where the epistemic and aleatory uncertainty coexists simultaneously. These R parameters are mathematically represented as

$$\begin{aligned} a_i &= a_{i,0}(1 + s_i\lambda_i); \quad N+1 \le i \le N+R \\ a_{\min,i} &\le a_{i,0} \le a_{\max,i}; \quad s_{\min,i} \le s_i \le s_{\max,i} \end{aligned} \tag{4}$$

where $[a_{\min,i}, a_{\max,i}]$ and $[s_{\min,i}, s_{\max,i}]$ represent the closed intervals of values the mean and relative SD can take. In such a scenario, a similar linear transformation as (3) can be used on the mean and relative SD quantities of (4) as

$$\begin{aligned} a_{i,0} &= \frac{(a_{\min,i} + a_{\max,i})}{2} + \frac{(a_{\max,i} - a_{\min,i})}{2}\mu_{i-N}; \\ s_i &= \frac{(s_{\min,i} + s_{\max,i})}{2} + \frac{(s_{\max,i} - s_{\min,i})}{2}\theta_{i-N} \end{aligned} \tag{5}$$

where $\mu_{i-N} \in [-1, 1]$ and $\theta_{i-N} \in [-1, 1]$. As before, there is no available information regarding the marginal PDF of μ_{i-N} and θ_{i-N} while the marginal PDF of λ_i is assumed to be known.

The overall vector of uncertain dimensions is $\boldsymbol{\alpha} = [\boldsymbol{\lambda}\ \boldsymbol{\beta}]$ where $\lambda = [\lambda_1, \lambda_2, \ldots, \lambda_{N+R}]$ is the vector of aleatory dimensions and $\boldsymbol{\beta} = [\boldsymbol{\xi\mu\theta}] = [\xi_1, \ldots, \xi_M, \mu_1, \ldots, \mu_R, \theta_1, \ldots, \theta_R]$ is the vector of epistemic dimensions. The aleatory dimensions occupy the $N + R$-dimensional space $\boldsymbol{\Omega}_a$ and the epistemic dimensions occupy the $M + 2R$-dimensional space $\boldsymbol{\Omega}_e$. It is assumed that all the network dimensions are uncorrelated. Finally, the total number of uncertain dimensions in (1) is $P = M+N+3R$. The impact of these uncertain P dimensions on any single network response $x(t, \boldsymbol{\alpha})$ can be quantified using the statistical moments of the response. The arbitrary jth statistical moment of the response $x(t, \boldsymbol{\alpha})$ is expressed as the integral

$$m_j(t, \boldsymbol{\beta}) = \int_{\Omega_a} (x(t, \boldsymbol{\lambda}, \boldsymbol{\beta}))^j \rho(\boldsymbol{\lambda}) d\boldsymbol{\lambda} \tag{6}$$

where $\rho(\boldsymbol{\lambda})$ is the joint PDF of the aleatory dimensions $\boldsymbol{\lambda}$. It is observed from (6) that the jth statistical moment is an integral over the entire aleatory domain $\boldsymbol{\Omega}_a$. As a result, the value of the statistical moments of the response is dependent on the values of the epistemic dimensions $\boldsymbol{\beta}$. The goal of MUQ is to evaluate the maximum and minimum bounds circumscribing this variability of any response statistics [5]–[7].

C. Existing Hybrid CPC Approach for MUQ

It is impossible to use conventional PC metamodels to capture the mixed uncertainty in a network response. This is because generalized PC theory dictates that the basis functions used must be mutually orthonormal with respect to the joint probability density function (PDF) of the input dimensions [25]. However, for a mixed uncertainty problem, the marginal PDFs of the epistemic dimensions, and consequently, the joint PDF of the input dimensions are always unknown. To resolve this issue, in recent works, a hybrid CPC metamodel has been developed for MUQ [22], [23]. The CPC metamodel represents the mixed uncertainty in a network response $x(t, \alpha)$ using the closed-form expansion

$$x(t, \boldsymbol{\alpha}) \approx \sum_{k=0}^{N_{PC}-1} x_k(t)\phi_k(\lambda)\psi_k(\boldsymbol{\beta}) \tag{7}$$

where $\phi_k(\boldsymbol{\lambda})$ is the kth multivariate PC basis function taken from the Wiener–Askey scheme [25], $\psi_k(\boldsymbol{\xi})$ is the kth multivariate Chebyshev basis function, and $x_k(t)$ is the corresponding coefficient. The number of terms in the expansion of (7) is equal to $N_{PC} = (P + d)!/(P!d!)$ where d is the maximum common degree of the PC and Chebyshev bases.

The rationale for using a CPC metamodel is that it leverages orthonormal basis functions to represent both epistemic and aleatory uncertainties even in the absence of any knowledge regarding the marginal PDFs of the epistemic network dimensions. In other words, the basis functions $\phi_k(\lambda)$ and $\psi_k(\xi)$ of (7) both satisfy the inner product relationship [22]

$$\begin{aligned} \langle \phi_j, \phi_k \rangle = \langle \psi_j, \psi_k \rangle &= \int_{\Omega_a} \phi_j(\boldsymbol{\lambda})\phi_k(\boldsymbol{\lambda})w(\boldsymbol{\lambda})d\boldsymbol{\lambda} \\ &= \int_{\Omega_e} \psi_j(\boldsymbol{\beta})\psi_k(\boldsymbol{\beta})\chi(\boldsymbol{\beta})d\boldsymbol{\beta} = \delta_{jk} \\ w(\boldsymbol{\lambda}) = \rho(\boldsymbol{\lambda});\ \chi(\boldsymbol{\beta}) &= \prod_{j=1}^{M+2R} \frac{1}{\sqrt{1-\beta_j^2}}. \end{aligned} \tag{8}$$

Note that the weight function $\chi(\boldsymbol{\beta})$ is specially selected in (8) to ensure that the bases $\psi_k(\boldsymbol{\xi})$ belong to the Chebyshev polynomial family, same as in [22]. Due to the orthonormality condition of (8) displayed by the basis functions of (7), the CPC approximation of (7) will quickly converge to the correct results over the multidimensional mixed uncertainty space $\boldsymbol{\Omega}_a \times \boldsymbol{\Omega}_e$ even though the marginal PDFs of epistemic dimensions remain unknown [22], [23]. The coefficients of (7) can be evaluated using any nonintrusive approach. Once the coefficients are known, the metamodel of (7) can be used as a closed-form proxy to the SPICE model of (1). Hence, it is drastically more efficient to use the closed-form metamodel of (7) instead of the large SPICE model of (1) to extract the maximum and minimum bounds of the statistics of $x(t, \boldsymbol{\alpha})$.

The main limitation of the CPC metamodel is that in order to evaluate the unknown coefficients of (7), the number of discrete SPICE simulations of the network of (1) that is required is $2N_{PC}$. In other words, the number of required SPICE simulations scale as $O(P^d) = O((N+M+3R)^d)$ with respect to the total number of dimensions P [16]. Thus, even for moderate values of N, M, and R, the number of required SPICE simulations will be computationally intractable. While various methods such as those based on ANOVA [16], [18], [27], [34], active subspaces [33], basis adaptations [28]–[31], truncated Karhunen–Loeve decomposition [32], and probabilistic error criteria [35] have been developed to reduce the dimensionality of the problem for purely aleatory problems, all of these methods invariably fail when applied to mixed uncertainty problems. This is because all of these methods rely on the statistical information of the response in order to determine how to best reduce the number of unknown coefficients in the metamodel. Unfortunately, as described in (6), the response statistics of mixed uncertainty problems vary with the local values of the epistemic dimensions $\boldsymbol{\beta}$. Because of this local variability in the response statistics, none of the aforementioned conventional approaches can extract any objective and unbiased information to guide the metamodel reduction process.

Among other techniques, methods based on l_0 norm regularization [36], sensitivity-based principal component analysis [37], and tensor-train decomposition [38] has been successfully deployed for high-dimensional circuit simulation problems. For example, the l_0 norm regularization approach finds the sparsest subset of PC bases by calculating an importance measure for each basis function. On the other hand, the work of [37] couples a principal component analysis scheme with a sensitivity analysis scheme to identify a subset of the most important uncorrelated dimensions that have maximum influence on the network response. Thereafter, a PC metamodel is constructed for this reduced subset of dimensions. Finally, the tensor train decomposition of [38] significantly accelerates the computation of statistical moments in massively high-dimensional spaces. Despite the various benefits of all these works, their extension to mixed problems remains nontrivial. This is because the epistemic dimensions $\boldsymbol{\beta}$ will still introduce local variability in the importance analysis of basis functions, the sensitivity analysis scheme, and the statistical moment calculations. Thus, irrespective of the approach chosen, reducing the CPC metamodel remains an open problem. This inability to reduce the CPC metamodel means that the massive $O(P^d)$ number of SPICE simulations required cannot be reduced in any way. As a result, MUQ of MTL networks with even a modest number of uncertain dimensions (P) is a challenge for circuit designers.

III. Development of Proposed Dimension Reduction Approach for Mixed Uncertainty Problems

In this section, the proposed sensitivity sweeping-based dimension reduction algorithm, developed especially for mixed uncertainty problems, is described. As a starting point of this algorithm, a robust numerical representation of the network response $x(t, \boldsymbol{\alpha})$, referred to as a forward model, needs to be constructed. This forward model is constructed in multiple stages, as described in Sections III-A–III-C.

A. Constructing the Initial Forward Model of the Response

In order to construct a forward model of any network response $x(t, \boldsymbol{\alpha})$ of (1), that response is first decomposed into a sum of hierarchical interaction terms as [39]

$$x(t, \boldsymbol{\alpha}) = x_0(t) + \sum_{i=1}^{P} x_i(t, \alpha_i) + \sum_{1\le i,j\le P}^{x_i(t,\alpha_i,\alpha_j)} + \cdots + x_{12\ldots P}(t, \boldsymbol{\alpha}) \quad (9)$$

where x_0 is the value of $x(t,\boldsymbol{\alpha})$ in the absence of any uncertainty (zeroth interaction term), $x_i(t,\alpha_i)$ represents the contribution of each α_i on $x(t,\boldsymbol{\alpha})$ acting alone (one-way interaction terms), $x_{ij}(t,\alpha_i,\alpha_j)$ represents the contribution of each $\{\alpha_i, \alpha_j\}$ pair on $x(t,\boldsymbol{\alpha})$ acting alone (two-way interaction terms) etc. The zeroth and one-way interaction terms of (9) are expressed as [39]

$$\begin{aligned} x_0(t) &= x(t, \boldsymbol{\alpha}^{(0)}) \\ x_i(t, \alpha_i) &= .x(t, \boldsymbol{\alpha})|_{\boldsymbol{\alpha}^{(0)}\backslash\alpha_i} - x_0(t); \quad 1 \le i \le P \end{aligned} \quad (10)$$

where $\boldsymbol{\alpha}^{(0)} = \mathbf{0}$ and the notation $\boldsymbol{\alpha}^{(0)}\backslash\alpha_i$ represents the vector where all components of $\boldsymbol{\alpha}$ except α_i are set to 0.

In the work of [26], a forward model of the response $x(t, \boldsymbol{\alpha})$ was approximated using only the zeroth and one-way interaction terms as

$$x(t, \boldsymbol{\alpha}) \approx x_0(t) + \sum_{i=1}^{P} x_i(t, \alpha_i). \quad (11)$$

However, this forward model is too simplistic and grossly inadequate for the scenario where epistemic and aleatory uncertainties simultaneously coexist within the same model parameters. To better understand why that is the case, assume $\alpha_i = \theta_{i-N}$. Now, when quantifying $x_i(t,\alpha_i)$, all component of $\boldsymbol{\alpha}$ except $\alpha_i = \theta_{i-N}$ are set to 0 as shown in (10). This means that the dimensions $\lambda_i = \mu_{i-N} = 0$. Replacing $\lambda_i = \mu_{i-N} = 0$ in (4) and (5) automatically leads to $x_i(t,\alpha_i) = 0$ in (10). Thus, based on the forward model of (11), the contribution of any epistemic dimension θ_{i-N} to any network response is always zero—a completely incorrect conclusion. This incorrect conclusion is an artifact of the forward model of (11) caused by the fact that all two-way or higher interaction terms in the forward model have been neglected even when the SD quantities, s_i in (4), have uncertainty embedded in them.

In order to eliminate this artifact from the forward model, in this paper, a new iterative algorithm to construct the forward model is proposed. In this algorithm, initially the forward model is assumed to contain all zeroth, one-way, and only a select few two-way interaction terms of (9). Specifically, the two-way interaction terms included are those where the pair $\{\alpha_i, \alpha_j\} = \{\lambda_i, \theta_{i-N}\}$ in (9). These specific two-way interaction terms are selected because they partially capture the contribution of each epistemic dimension θ_{i-N} which was not possible by simply using the one-way interaction terms in (11). Next, the initial forward model is mathematically represented using a CPC-based metamodel as

$$\begin{aligned} x(t, \boldsymbol{\alpha}) &\approx x_0(t) + \sum_{i=1}^{P} x_i(t, \alpha_i) + \sum_{1\le i,j\le P} x_i(t, \alpha_i, \alpha_j) \\ &= \sum_{i=1}^{N+R}\sum_{r=0}^{d} x_i^{(r)}(t)\phi_r(\lambda_i) + \sum_{i=1}^{M+2R}\sum_{r=0}^{d} y_i^{(r)}(t)\psi_r(\beta_i) \\ &\quad + \sum_{i=1}^{R}\sum_{j=2}^{d}\sum_{r=1}^{j-1} z_i^{(r)}(t)\phi_r(\lambda_{i+N})\psi_{d-r}(\theta_i) \end{aligned} \quad (12)$$

where $x_i^r(t)$, $y_i^r(t)$, and $z_i^r(t)$ are the coefficients representing the contributions of the aleatory, epistemic, and the two-way mixed dimension interactions, respectively. Now, replacing the functionality of the zeroth and one-way interaction terms of (10) in (12) allows the reformulation of (12) as an $\boldsymbol{Ax} = \boldsymbol{B}$ over-determined linear system of equations for N_{sim} nodes which can be solved in a least-squares manner to evaluate the coefficients of (12) similar to what has been done for purely aleatory problems in [15]. The CPC metamodel representing the forward model in (12) is hereafter referred to as $F_1(t, \boldsymbol{\alpha})$. $F_1(t, \boldsymbol{\alpha})$ is relatively sparse compared to the full-blown CPC metamodel of (7) because it only includes a small number of two-way interaction terms and nothing else. In fact, the number of unknown coefficients (N_{co}) and the corresponding number of SPICE simulations required (N_{sim}) to evaluate the coefficients of $F_1(t, \boldsymbol{\alpha})$ in (12) is limited to

$$N_{\text{co}} = \frac{d(d-1)R}{2} + (d+1)(M+N+3R); \quad N_{\text{sim}} = 2N_{\text{co}} \quad (13)$$

where $N_{\text{co}} \ll N_{\text{PC}}$ and consequently $N_{\text{sim}} \ll 2N_{\text{PC}}$. Next, the forward model $F_1(t, \boldsymbol{\alpha})$ is used to calculate the sensitivity of the response variance with respect to each epistemic and aleatory dimension of the network. This sensitivity analysis is based on a sensitivity sweeping algorithm.

B. Sensitivity Analysis Using a Sensitivity Sweeping Algorithm

The sensitivity analysis of the network dimensions is broken into two parts. In the first part, the sensitivity of the response variance with respect to all aleatory network dimensions is quantified. This is followed by the second part where the sensitivity of the response variance with respect to all epistemic network dimensions is quantified. Based on these sensitivity analyses, the network dimensions are ranked in order of their importance.

The main challenge in estimating the sensitivity of the response variance with respect to any network dimension is that the variance itself varies with the values of the epistemic dimensions $\boldsymbol{\beta}$. This variability in the response variance means that the sensitivity information of any dimension also varies with the values of the epistemic dimensions $\boldsymbol{\beta}$. In order to capture the variability of the sensitivity information of any dimension over the entire epistemic space, a sensitivity sweeping algorithm is proposed. This algorithm is based on a nested sampling approach. As part of this nested sampling approach, in the outer loop, N_e uniformly distributed sample points or nodes are selected spanning the entire epistemic space Ω_e without replacement. Note that in the absence of any knowledge regarding the PDF of the epistemic dimensions, a choice of uniformly distributed sampling nodes is the most appropriate. Let any pth node be described as $\boldsymbol{\beta}^p = [\xi_1^p, \ldots, \xi_M^p, \mu_1^p, \ldots, \mu_R^p, \theta_1^p, \ldots, \theta_R^p]$ where $1 \leq p \leq N_e$. Replacing each node $\boldsymbol{\beta}^p$ in the forward model $F_1(t, \boldsymbol{\alpha})$ of (12) renders the model as a pure PC metamodel of the aleatory dimensions λ as

$$F_1(t, \boldsymbol{\lambda}, \boldsymbol{\beta} = \boldsymbol{\beta}^{(p)}) \approx \hat{x}_0(t) + \sum_{i=1}^{N+R} \sum_{r=1}^{d} \hat{x}_i^{(r)}(t)\phi_r(\lambda_i) \quad (14)$$

where $\hat{x}_0$ represents the mean and $\hat{x}_i^{(r)}$ are the rth PC coefficients capturing the impact of λ_i on the response, all computed at the node location $\boldsymbol{\beta} = \boldsymbol{\beta}^p$. The term $\hat{x}_0$ and coefficients $\hat{x}_i^{(r)}$ vary with the values of the epistemic dimensions $\boldsymbol{\beta}$ and hence will change for each node $\boldsymbol{\beta}^p$. Now, in the inner loop, the first-order Sobol's sensitivity index of the aleatory dimension λ_1 is computed as [40]

$$A_1(t, \boldsymbol{\beta} = \boldsymbol{\beta}^{(p)}) = \frac{V(t, \boldsymbol{\beta}^{(p)})}{V_u(t, \boldsymbol{\beta}^{(p)})} = \frac{\sum_{r=1}^{d} \left(\hat{x}_1^{(r)}(t)\right)^2}{\sum_{i=1}^{N+R} \sum_{r=1}^{d} \left(\hat{x}_i^{(r)}(t)\right)^2} \quad (15)$$

where $V(t, \boldsymbol{\beta}^p)$ is the contribution of the aleatory dimension λ_1 to the response variance and $V_u(t, \boldsymbol{\beta}^p)$ is the overall response variance, both estimated at the node location $\boldsymbol{\beta} = \boldsymbol{\beta}^p$. The index of (15) is further reduced to a scalar quantity by integrating the numerator and denominator of (15) over the entire time window of simulation, $[0 - T_{\max}]$ as shown

$$\hat{A}_1(\boldsymbol{\beta} = \boldsymbol{\beta}^{(p)}) = \frac{\int_0^{T_{\max}} \sum_{r=1}^{d} \left(\hat{x}_1^{(r)}(t)\right)^2 dt}{\int_0^{T_{\max}} \sum_{i=1}^{N+R} \sum_{r=1}^{d} \left(\hat{x}_i^{(r)}(t)\right)^2 dt}. \quad (16)$$

The integrals of (16) can be evaluated using any appropriate numerical integration rule [41].

This process of evaluating the local sensitivity index of λ_1 at $\boldsymbol{\beta} = \boldsymbol{\beta}^p$, performed inside the inner loop, is next repeated for all N_e nodes of the outer loop. In so doing, the scalar sensitivity index of (15) is evaluated at all the N_e nodes spanning the entire epistemic space Ω_e. Thus, the index of (16) is considered to have been "swept" across the entire epistemic space Ω_e in a point-by-point manner. This is why the above algorithm is referred to as a sensitivity sweeping algorithm. The above sensitivity sweeping algorithm is then repeated for all $N + R$ aleatory dimensions. From the ensemble of the sensitivity indices of (16), the relative importance of λ_1 with respect to all other aleatory dimensions is expressed as the ratio

$$I_{\lambda_1} = \frac{\sum_{p=1}^{N_e} \hat{A}_1(\boldsymbol{\beta} = \boldsymbol{\beta}^{(p)})}{\sum_{q=1}^{N+R} \sum_{p=1}^{N_e} \hat{A}_q(\boldsymbol{\beta} = \boldsymbol{\beta}^{(p)})}. \quad (17)$$

It is pointed out that the relative importance measure of (17) takes into account the variability of the response variance across the entire epistemic space. As a result, this measure represents the correct relative sensitivity of each aleatory dimension that could not otherwise be determined using conventional means due to the variability of the response variance.

In the above algorithm, a total of N_e response variances and $(N + R)N_e$ Sobol's sensitivity indices of (16) are quantified. All of these computations does not involve any SPICE simulations and are done analytically using the forward model $F_1(t, \boldsymbol{\alpha})$. Moreover, all of these computations can be easily performed in parallel. Thus, the CPU expense of the sensitivity sweeping algorithm when applied for aleatory dimensions is negligible.

Next, the sensitivity analysis for the epistemic dimensions is performed. At this juncture, it is strongly emphasized that currently there is no known approach to estimate the sensitivity indices of epistemic dimensions of a mixed problem. In this paper, this limitation is resolved by observing that the response variance $V_u(t, \boldsymbol{\beta}^p)$ of (15) has already been evaluated at all N_e nodes. This ensemble of variance measures are referred to as unconditional variances. From these N_e unconditional variances, the unconditional maximum and minimum bounds of the variance are determined as

$$V_{\min}(t) = \min(V_u(t, \boldsymbol{\beta}^{(p)}),\ 1 \le p \le N_e);\\ V_{\max}(t) = \max(V_u(t, \boldsymbol{\beta}^{(p)}),\ 1 \le p \le N_e). \tag{18}$$

Thereafter, the response variances at N_e sample nodes are revaluated subject to the constraint that at all N_e nodes, the value $\beta_1^p = \beta_1^{(1)}$. In other words, a new ensemble of variance measures are determined subject to the condition that the first epistemic dimension β_1 is fixed at the location $\beta_1^{(1)}$. The corresponding variance bounds are called the conditional maximum and minimum bounds. Mathematically, the conditional bounds can be described as

$$V_{\min,1,1}(t) = \min(V_u(t, \boldsymbol{\beta}^{(p)}) \forall \beta_1^{(p)} = \beta_1^{(1)}, 1 \le p \le N_e);\\ V_{\max,1,1}(t) = \max(V_u(t, \boldsymbol{\beta}^{(p)}) \forall \beta_1^{(p)} = \beta_1^{(1)}, 1 \le p \le N_e). \tag{19}$$

From the knowledge of the unconditional and conditional bounds, the area enclosed within these bounds across the entire time window of simulation is measured as

$$E_1 = \frac{1}{2\,T_{\max}} \left(\int_0^{T_{\max}} \left(V_{\max}(t) - V_{\max,1,1}(t) \right) dt - \int_0^{T_{\max}} \left(V_{\min}(t) - V_{\min,1,1}(t) \right) dt \right). \tag{20}$$

As before, the integral of (20) is evaluated using any appropriate numerical integration rule. The integral of (20) measures the change in the variance bounds caused by fixing β_1 at the location $\beta_1^{(1)}$. In other words, this integral measures the sensitivity of the variance bounds with respect to the epistemic dimension β_1 at the location $\beta_1 = \beta_1^{(1)}$. Thus, the integral of (20) is hereafter referred to as the local sensitivity index of β_1. The local sensitivity index of β_1 is then evaluated at all $\beta_1 = \beta_1^p$, $1 \le p \le N_e$ nodes. In this way, the sensitivity index of β_1 is "swept" across the entire 1-D space $[-1, 1]$ in which the dimension resides. This sensitivity sweeping algorithm is repeated for all $M + 2R$ epistemic dimensions of the network. To collate the information from the sensitivity sweeping, the relative importance of β_1 with respect to all other epistemic dimensions is expressed as the ratio

$$I_{\xi_1} = \frac{\sum_{p=1}^{N_e} E_1\left(\beta_1 = \beta_1^{(p)}\right)}{\sum_{q=1}^{M+2R} \sum_{p=1}^{N_e} E_q\left(\beta_1 = \beta_1^{(p)}\right)}. \tag{21}$$

Once again, the relative importance measure of (21) takes into account the variability of the response variance across the entire multidimensional epistemic space. Thus, this measure too represents the correct relative sensitivity of each epistemic dimension that could not otherwise be determined using conventional means due to the variability of the response variance.

It is observed that for the sensitivity analysis of the epistemic dimensions, N_e instances of the variances have to be computed to find out the variance bounds of (18). This process has to be repeated for a maximum of N_e sample points and $M + 2R$ dimensions. In other words, a maximum of $(M + 2R)N_e^2$ variances have to be computed. However, as explained before, these variance computations are done analytically using the forward model $F_1(t, \boldsymbol{\alpha})$ and can even be parallelized to ensure that negligible CPU costs are incurred.

To summarize, the proposed sensitivity sweeping algorithm is able to quantify the relative importance measure of all network dimensions, whether aleatory or epistemic, using (17) and (21) even in the presence of locally variable response variances—something that has hitherto not been possible to do. As a result, the measures of (17) and (21) can be directly used to rank the P network dimensions in order of their relative importance. From this ranking, the least important network dimensions can be identified.

C. Iteratively Enriching the Forward Model

In Section III-A, a methodology to construct a very sparse forward model $F_1(t, \boldsymbol{\alpha})$ was developed. While the sparsity of $F_1(t, \boldsymbol{\alpha})$ ensures that very few SPICE simulations of the original network is incurred in constructing the forward model, it also means that $F_1(t, \boldsymbol{\alpha})$ may not be accurate enough to correctly evaluate the relative importance of each network dimension in (17) and (21). This may lead to an incorrect ranking, and consequently, an incorrect identification of the least important network dimensions. To avoid this issue, the forward model $F_1(t, \boldsymbol{\alpha})$ needs to be sufficiently enriched by adding the remaining two-way interaction terms, and if needed, even the higher interaction terms of (9). In this section, an iterative algorithm to do so is described.

In order to iteratively enrich the forward model $F_1(t, \boldsymbol{\alpha})$, the entire set of P network dimensions are classified into three subsets—S_1 contains the critical network dimensions whose relative importance measure of (17) and (21) lies in the 75%–100% range, S_2 contains the moderately important network dimensions whose relative importance measure of (17) and (21) lies in the 1%–75% range, and S_3 contains the presumptive least important network dimensions whose relative importance measure of (17) and (21) is below 1%. Now, the forward model $F_1(t, \boldsymbol{\alpha})$ is enriched using the subsets S_1–S_3 as follows.

Step 1: The two-way interaction terms of the critical dimensions in subset $S = S_1$ need to be added first. The sum of these specific two-way interaction terms minus those two-way interaction terms already included in (12) are quantified as [39]

$$\sum_{\{\alpha_i,\, \alpha_j\} \subseteq S_1} x_{ij}(t, \alpha_i, \alpha_j) = x(t, \boldsymbol{\alpha})|_{\boldsymbol{\alpha}^{(0)} \setminus S_1} - \eta(S_1) x_0(t) - \sum_{\{\alpha_i,\, \alpha_j\} \subseteq S_1} (x_i(t, \alpha_i) + x_j(t, \alpha_j)) \tag{22}$$

where $\eta(S_1)$ is the cardinal number of the set S_1 minus the number of pairs $\{\alpha_i, \alpha_j\} = \{\lambda_i, \theta_{i-N}\}$ already considered in (12). The interaction terms of (22) are mathematically represented using a bunch of CPC metamodel terms, similar to (12), as

$$x(t, \boldsymbol{\alpha})|_{\boldsymbol{\alpha}^{(0)} \setminus S_1} - \eta(S_1) x_0(t) - \sum_{\{\alpha_i, \alpha_j\} \subseteq S_1} (x_i(t, \alpha_i) + x_j(t, \alpha_j)) \approx \sum_{\{\alpha_i, \alpha_j\} \subseteq S_1} y_k(t) \nu(\alpha_i) \kappa(\alpha_j) \tag{23}$$

where depending on whether α_i and α_j are epistemic or aleatory dimensions, $\nu(\alpha_i)$ and $\kappa(\alpha_j)$ are either Chebyshev or PC bases, respectively. As before, the coefficients of (23) are evaluated using linear regression. It is noted that evaluating the coefficients of (23) will automatically increase the number of nonintrusive SPICE simulations required (N_{sim}) beyond the value of $2N_{co}$ of (13).

Step 2: The CPC terms of (23) is added to the forward model $F_1(t, \boldsymbol{\alpha})$ so as to enrich its information content. The new and enriched forward model is called $F_2(t, \boldsymbol{\alpha})$ where

$$F_2(t, \boldsymbol{\lambda}, \boldsymbol{\xi}) = F_1(t, \boldsymbol{\lambda}, \boldsymbol{\xi}) + \sum_{\{\alpha_i,\, \alpha_j\} \subseteq S_1} y_k(t)\nu(\alpha_i)\kappa(\alpha_j). \quad (24)$$

At this point, it is important to check if $F_2(t, \boldsymbol{\alpha})$ is sufficiently rich in information or not. This check is done by repeating the sensitivity sweeping algorithm of Section III-B using $F_2(t, \boldsymbol{\alpha})$ instead of $F_1(t, \boldsymbol{\alpha})$. If the resultant change in the relative importance measure of (17) and (21) for all network dimensions is below 0.1%, then it is concluded that $F_2(t, \boldsymbol{\alpha})$ is already sufficiently rich in information. At this point, the dimensions in S_3 are the correct set of least important dimensions. Thus, the value of these dimensions are set to zero (i.e., removed from the network) and thereafter the reduced dimensional CPC metamodel is constructed as described in Section III-D. On the other hand, if even for a single network dimension, the resultant change in the relative importance measure of (17) or (21) is more than 0.1%, then it is concluded that $F_2(t, \boldsymbol{\alpha})$ is not sufficiently accurate. In that case, the contents of subsets S_1–S_3 are updated and the process moves on to step 3.

Step 3: Steps 1 and 2 are repeated where this time the subset of dimensions whose two-way interaction terms are added in (24) is $S = S_2$. This enriches the forward model $F_2(t, \boldsymbol{\alpha})$ to $F_3(t, \boldsymbol{\alpha})$. Now, if in repeating step 2, it is found out that $F_3(t, \boldsymbol{\alpha})$ is not sufficiently rich in information, then steps 1 and 2 have to be repeated, only this time the $\{\alpha_i, \alpha_j\}$ pair must satisfy $\alpha_i \subseteq S_1$ and $\alpha_j \subseteq S_2$.

Steps 1–3 enrich the forward model $F_1(t, \boldsymbol{\alpha})$ by adding the most important two-way interaction terms till the correct set of least important network dimensions are identified. By removing the least important network dimensions, it is possible to reduce the original P network dimensions to a much smaller number s. On the other hand, if adding all the two-way interaction terms still fails to sufficiently enrich the forward model, then steps 1–3 can be repeated, this time with the higher interactions terms of (9) included in (23) and (24).

D. Priority-Based Linear Regression Node Selection

Once the s reduced dimensions are identified, a new and reduced CPC metamodel is constructed similar to (7) with the difference being that the number of terms in the reduced metamodel is equal to $N_{\mathrm{RPC}} = (s + d)!/(s!d!)$ where $N_{\mathrm{RPC}} \ll N_{\mathrm{PC}}$. The coefficients of this reduced CPC metamodel are evaluated using the linear regression scheme of [15]. In this linear regression scheme, there are a total of $(d + 1)^s$ Gaussian quadrature nodes. Of these quadrature nodes, $N_{\mathrm{reg}} = 2N_{\mathrm{RPC}}$ nodes are required where the nonintrusive SPICE simulations of the network have to be performed. It is pointed out that these new SPICE simulations are in addition to the N_{sim} SPICE simulations already required to evaluate the coefficients of the forward model in (12) and (23). Therefore, the N_{sim} SPICE simulations to evaluate the coefficients of the forward model in (12) and (23) appear as CPU overheads. One method to reduce these overheads is by using steps 1–3 laid out in Section III-D. Note that in these steps, the interaction terms of S_1 and S_2 are not included all together but rather in sequence. This is done deliberately so that if the forward model is sufficiently enriched after adding the interaction terms of S_1, then adding the interaction terms of S_2 can be avoided. This allows the user to control the N_{sim} number of SPICE simulations required to fashion the forward model.

In this section, a new priority-based node selection strategy to even further reduce the CPU overheads is proposed. As part of this strategy, the overall set of $(d+1)^s$ Gaussian quadrature nodes is split into two sets, ζ and π. The set ζ holds the nodes where the N_{sim} SPICE simulations of the network have already been performed in order to evaluate the coefficients of the forward model in (12) and (23). The set π holds the remaining $(d+1)^s - \eta(\zeta)$ nodes where η represents the cardinal number of set ζ. Now, when searching for each N_{reg} regression node, first all nodes in the set ζ are checked to see if they can be reused based on the D-optimality criterion [15]. Only if no suitable node is found in the set ζ does the search proceed to set π. In effect, a higher priority is bestowed on the nodal set of ζ over π. The rationale behind this priority based node selection is to reuse as many of the N_{sim} simulation results in the linear regression scheme. In this way, only a fraction of the new N_{reg} SPICE simulations will have to be performed to evaluate the coefficients of the reduced dimensional CPC metamodel.

Assuming that ω fraction of the nodes from the set ζ are reused, the resultant efficiency in terms of the number of SPICE simulations required achieved by the proposed dimension reduction scheme over the full-blown CPC metamodel of (7) is of the order

$$\frac{2N_{\mathrm{PC}}}{N_{\mathrm{reg}} + (1-\omega)\eta(\boldsymbol{\zeta})} = \frac{2P^d + O(2P^{d-1})}{2s^d + (1-\omega)\eta(\boldsymbol{\zeta}) + O(2s^{d-1})} \approx \frac{2P^d}{2s^d + (1-\omega)\eta(\boldsymbol{\zeta})}. \quad (25)$$

From (25), it is inferred that the maximum efficiency bound of the reduced dimensional CPC metamodel over the full-blown CPC metamodel is of the order $O((P/s)^d)$. This occurs when all the nodes of set ζ are reused (i.e., $\omega = 1$).

IV. Numerical Examples

In this section, three examples are presented to demonstrate the accuracy and efficiency of the proposed reduced dimensional CPC metamodel over the full-blown CPC metamodel of [22], [23]. All relevant metamodel coefficient computations are performed using MATLAB 2013b while the deterministic transient simulations are performed using HSPICE [42]. In particular, for all examples, the MTL networks are modeled using the W-element transmission line model provided by

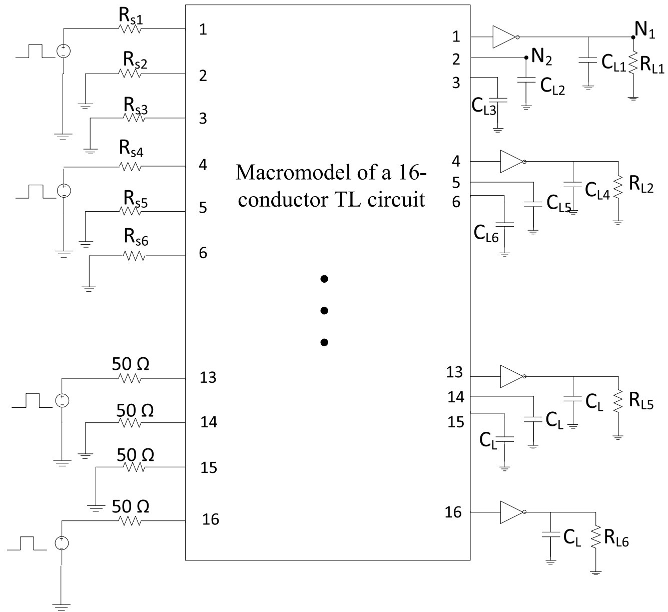


Fig. 1. Schematic of the MTL network of Example 1.

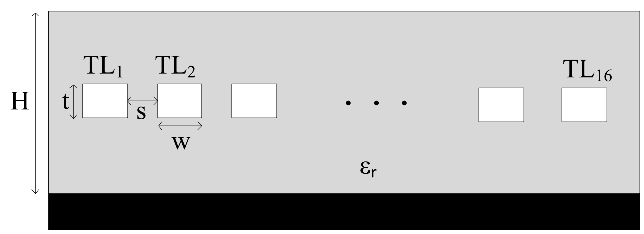


Fig. 2. Cross-sectional view of the MTL network of Fig. 1.

HSPICE which by default includes frequency-dependent per-unit-length parameters [42].

Example 1: In this example, the 16 conductor stripline MTL networks of Fig. 1 terminated by inverters consisting of SPICE level 49 CMOS transistors is considered. The response of interest for this example is the transient voltage at node N_1 of Fig. 1. The cross-sectional layout and geometric dimensions of the transmission lines are shown in Fig. 2. The voltage sources of Fig. 1 exhibit a trapezoidal waveform of rise/fall time $T_r = 0.1$ ns, pulsewidth $T_w = 5$ ns, and amplitude of 5 V. All the uncertain parameters of this example possess normal aleatory PDFs and are listed in Table I. The total number of network dimensions are $P = 36$.

In order to demonstrate the accuracy of the proposed approach, the uncertainty quantification for this example is performed using three metamodels—the conventional full-blown CPC metamodel [22], [23], the author's previous work of [26], and the proposed reduced dimensional CPC metamodel where the degree of expansion of all metamodels are set to $d = 3$. In constructing the reduced dimensional metamodel, the number of epistemic samples used in the sensitivity sweeping algorithm is set to $N_e = 10\,000$. In addition, including all two-way interactions between dimensions in S_1 makes the forward model sufficiently enriched to correctly identify the unimportant network dimensions of this example. The number of SPICE simulations incurred in constructing the forward model is $N_{\text{sim}} = 248$. Using the forward model, the original $P = 36$ dimensions of the network is reduced to $s = 8$ dimensions. In particular, all mixed dimensions of parameter L, the epistemic mean and aleatory dimensions of parameters w and s, and the epistemic mean of resistance R_{s1} are retained. All other dimensions are discarded. It is noted that this reduction is substantially greater than the reduction from $P = 36$ to $s = 30$ achieved by the work of [26]. This greater reduction is possible because this paper reduces both the epistemic and aleatory network dimensions unlike the work of [26] which only reduces the aleatory network dimensions.

TABLE I

UNCERTAIN PARAMETERS OF EXAMPLE 1 (FIG. 1)

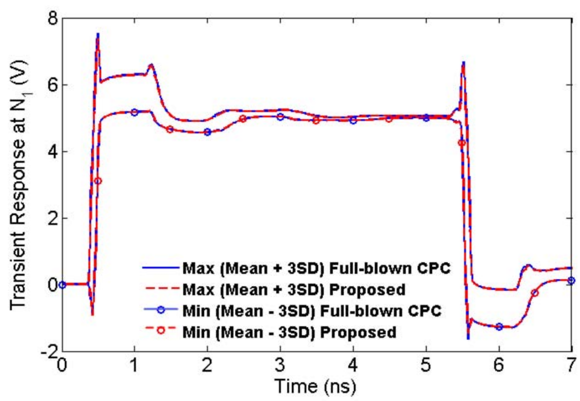
Random Variable	Mean ($a_{i,0}$)	Relative Standard Deviation (s_i)
w	[142.5 – 157.5] μm	[9.5 – 10.5] %
s	[142.5 – 157.5] μm	
PL (PMOS length)	[0.095 – 0.105] μm	
PW (PMOS width)	[9.5 - 10.5] μm	
NL (NMOS length)	[0.095 – 0.105] μm	
NW (NMOS width)	[9.5 - 10.5 μm	
σ (TL conductance)	[5.51 – 6.09] e7	
t	[27.5 – 32.5] μm	
H	[427.5 – 495] μm	
ε_r	[3.89 – 4.30]	
R_{s1}	[1.35 – 1.65] kΩ	
L (TL length)	[5.7 - 6.3] cm	[4.75 – 5.25] %

Next, the maximum bound of mean plus three times the SD and minimum bound of mean minus three times the SD of the transient response at node N_1 is evaluated using the aforementioned three methods. The reason these particular statistical results are chosen is because they represent the maximum and minimum possible probabilistic bounds of the transient response at node N_1 caused due to the mixed uncertainty. These results are compared in Fig. 3(a). Furthermore, in order to demonstrate the accuracy of the proposed approach for higher order statistical moments, the bounds of the cumulative density function (CDF) of the transient response at N_1 is evaluated at the time point when the mean is maximum ($t = 1.1$ ns) using the full-blown and the proposed reduced dimensional CPC metamodels. The maximum bound of the CDF is called the plausibility function and the minimum bound is called the belief function. These results are compared in Fig. 3(b). From Fig. 3(a) and (b), it can be concluded that the reduced dimensional CPC metamodel shows good agreement with its full-blown counterpart.

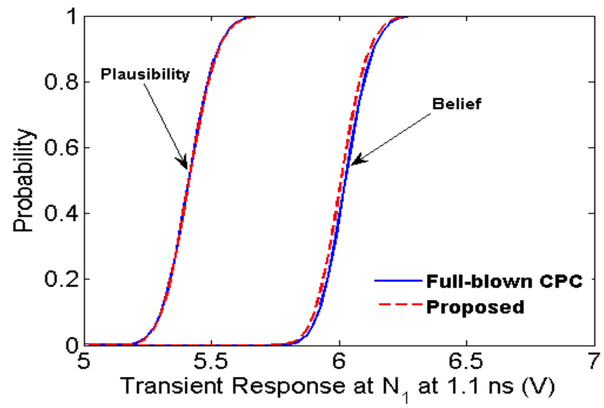
The total number of SPICE simulations incurred by the full-blown CPC metamodel, the work of [26], and the proposed reduced dimensional CPC metamodel are reported in Table II. It is observed from Table II that using the priority-based node selection scheme improves the speedup of the proposed metamodel over the full blown CPC metamodel and the work of [26] from roughly 32 and 19 times, respectively, to 55 and 33 times, respectively. For this example, the full set of $N_{\text{sim}} = 248$ SPICE simulations required to construct the forward model is reused in the priority based node selection scheme.

TABLE II
CPU COST ANALYSIS EXAMPLE 1

Approach	Original # Dimensions (P)	Reduced # Dimensions (p)	# SPICE Simulations	CPU cost (in minutes)	Speedup
Full-blown CPC Metamodel	36	-	18,278	82.35	-
Work of [27]		30	11,057	49.76	1.65
Proposed reduced dimensional metamodel (no priority based node selection)		8	578	2.60	31.67
Proposed reduced dimensional metamodel (using priority based node selection)		8	330	1.49	55.27



(a)



(b)

Fig. 3. Statistics of the transient response at node N_1 of Fig. 1. (a) Maximum bound of mean plus three times the SD and minimum bound of mean minus three times the SD of the transient response at node N_1. (b) Belief and plausibility CDFs of the transient response at node N_1 at time point when mean of the response is maximum (i.e., at $t = 1.1$ ns).

Example 2: In this example, the same MTL network of Fig. 1 is considered with the responses of interest being the transient voltage at both nodes N_1 and N_2. The uncertainty in the network is introduced using $P = 45$ parameters, the characteristics of which are listed in Table III.

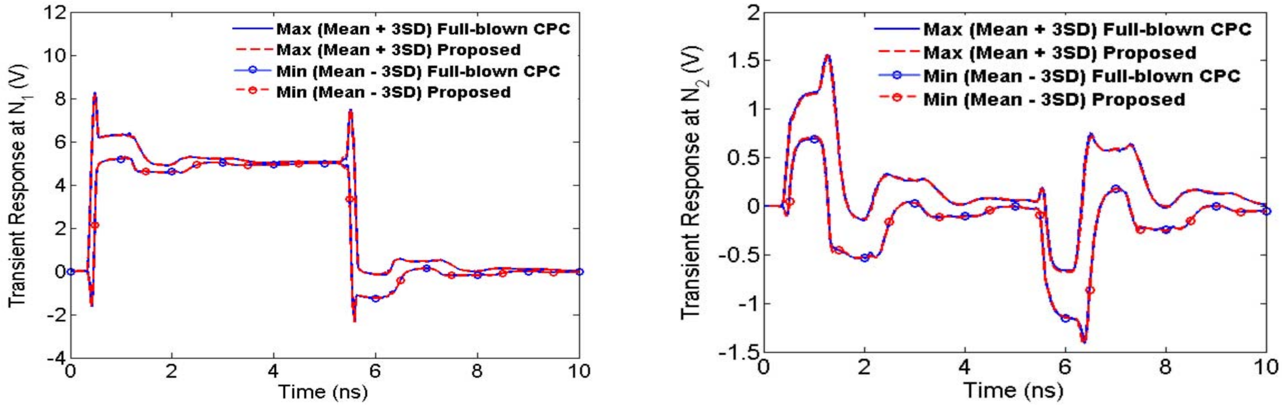
The uncertainty quantification for this example is performed using the same three metamodels as in Example 1. When constructing the reduced dimensional metamodel, the number of epistemic samples used in the sensitivity sweeping algorithm is still kept at $N_e = 10\,000$. Just as in Example 1, including all two-way interactions between dimensions in S_1 makes the forward model of this example sufficiently enriched to correctly identify the unimportant network dimensions. The number of SPICE simulations incurred in constructing the forward model is $N_{\text{sim}} = 284$. Leveraging this forward model, the original $P = 45$ mixed dimensions is reduced to $s = 12$ dimensions. In particular, all mixed dimensions of parameters T and ε_r, the epistemic mean and aleatory dimensions of parameters w and s, and the aleatory dimensions of capacitances C_{L1} and C_{L2} are retained. All other dimensions are discarded. As expected, this reduction is substantially greater than the reduction from $P = 45$ to $s = 35$ achieved by the work of [26].

The same statistical bounds of Example 1 are evaluated for the transient responses at nodes N_1 and N_2 using the aforementioned methods and the results are compared

TABLE III
UNCERTAIN PARAMETERS OF EXAMPLE 2 (FIG. 1)

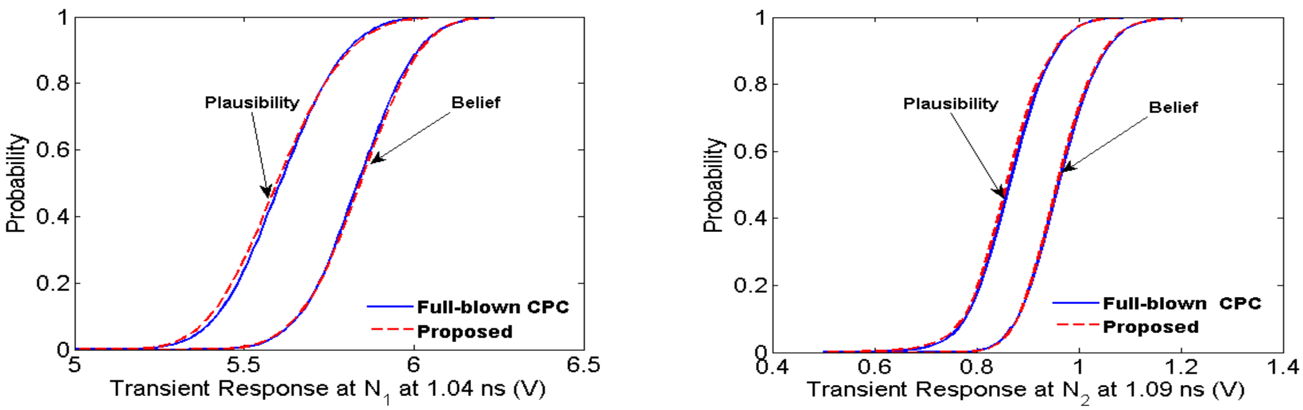
Random Variable	Mean ($a_{i,0}$)	Relative Standard Deviation (s_i)
w	[147.5 – 152.5] μm	[9.5 – 10.5] %
s		
R_{L1}	[1.425 – 1.575] kΩ	
R_{L2}		
R_{L3}		
PL (PMOS length)	[0.095 – 0.105] μm	
PW (PMOS width)	[9.5 – 10.5] μm	
NL (NMOS length)	[0.095 – 0.105] μm	
NW (NMOS width)	[9.5 – 10.5] μm	
t	[28.5 – 31.5] μm	
H	[427.5 – 472.5] μm	
C_{L1}	[0.95 – 1.05] pF	
C_{L2}		
C_{L3}		
ε_r	[3.895 – 4.305]	

in Fig. 4. Furthermore, in order to demonstrate the accuracy of the proposed approach for higher order statistical moments, the CDF bounds of the transient responses at N_1 and N_2 are evaluated at the time points when the corresponding means are maximum ($t = 1.04$ ns and $t = 1.09$ ns, respectively) using the same aforementioned methods. These results are compared



(a) (b)

Fig. 4. Statistics of the transient responses at nodes N_1 and N_2 of Fig. 1. (a) Maximum bound of mean plus three times the SD and minimum bound of mean minus three times the SD of the transient response at node N_1. (b) Maximum bound of mean plus three times the SD and minimum bound of mean minus three times the SD of the transient response at node N_2.



(a) (b)

Fig. 5. Maximum and minimum bounds of the CDFs of the transient responses at nodes N_1 and N_2 of Fig. 1. (a) Belief and plausibility CDFs of the transient response at node N_1 at time point when mean of the response is maximum (i.e., at $t = 1.04$ ns). (b) Belief and plausibility CDFs of the transient response at node N_2 at time point when mean of the response is maximum (i.e., at $t = 1.09$ ns).

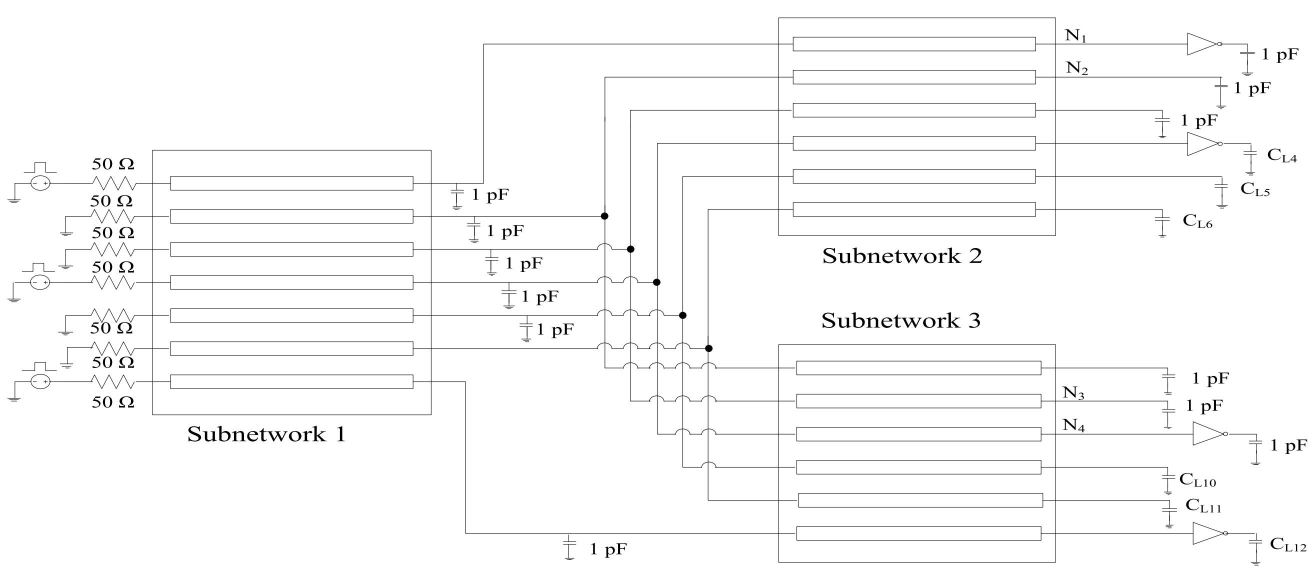
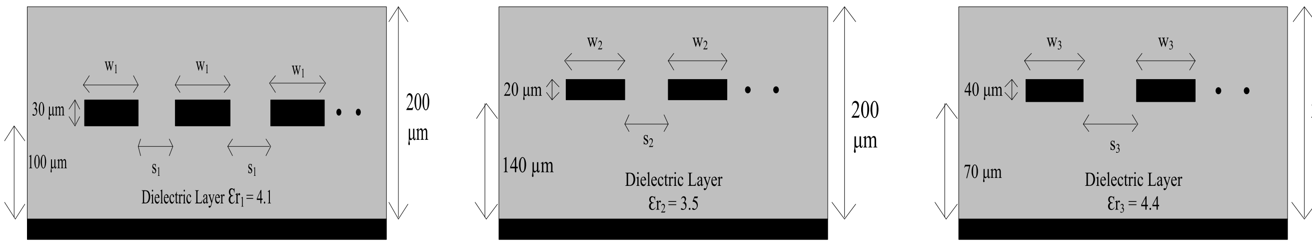


Fig. 6. Schematic of the MTL network of Example 3.

in Fig. 5. From Figs. 4 and 5, it is concluded that the reduced dimensional CPC metamodel shows good agreement with the full-blown CPC metamodel.

The total number of SPICE simulations incurred by the full-blown CPC metamodel, the work of [26], and the proposed reduced dimensional CPC metamodel are reported in Table IV.



(a) (b) (c)

Fig. 7. Cross-sectional view of the MTL network of Example 3. (a) Subnetwork 1. (b) Subnetwork 2. (c) Subnetwork 3.

TABLE IV
CPU COST ANALYSIS EXAMPLE 2

Approach	Original # Dimensions (P)	Reduced # Dimensions (p)	# SPICE Simulations	CPU cost (in minutes)	Speedup
Full-blown CPC Metamodel	45	-	34,592	155.66	-
Work of [27]		35	17,053	76.74	2.02
Proposed reduced dimensional metamodel (no priority based node selection)		12	1,194	5.37	28.98
Proposed reduced dimensional metamodel (using priority based node selection)		12	910	4.10	37.97

It is observed from Table IV that using the priority-based node selection scheme improves the speedup of the proposed metamodel over the full blown CPC metamodel and the work of [26] from approximately 29 and 14 times, respectively, to 38 and 19 times, respectively. For this example, the full set of $N_{\text{sim}} = 284$ SPICE simulations are reused in the priority based node selection scheme.

Example 3: In this example, the MTL network of Fig. 6 terminated by inverters consisting of SPICE level 49 CMOS transistors is considered. The cross-sectional layout and geometric dimensions of the MTL subnetworks are shown in Fig. 7. The lengths of the transmission lines in subnetworks 1, 2, and 3 are set to 5, 10, and 15 cm, respectively. The voltage sources of Fig. 6 exhibit a trapezoidal waveform of rise/fall time $T_r = 0.1$ ns, pulsewidth $T_w = 5$ ns, and amplitude of 5 V. The transient voltages at the nodes N_1, N_2, N_3, and N_4 are considered to be the responses of interest. All the uncertain parameters of this example possess normal aleatory PDFs and are listed in Table V. The total number of network dimensions are $P = 60$.

In order to demonstrate the accuracy of the proposed approach, the uncertainty quantification for this example is performed using the same metamodels of Example 1 where the degree of expansion of all the metamodels are set to $d = 3$. For this example, the sensitivity sweeping algorithm requires $N_e = 10\,000$ epistemic samples. As in Examples 1 and 2, including all two-way interactions between dimensions in S_1 makes the forward model of this example sufficiently enriched to correctly identify the unimportant network dimensions.

TABLE V
UNCERTAIN PARAMETERS OF EXAMPLE 3 (FIG. 6)

Random Variable	Mean ($a_{i,0}$)	Relative Standard Deviation (s_i)
w_1	[142.5 – 157.5] µm	[9.5 – 10.5] %
w_2	[123.5 – 136.5] µm	
w_3	[161.5 – 178.5] µm	
s_1	[95 – 105] µm	
s_2	[142.5 – 157.5] µm	
s_3	[190 – 210] µm	
C_{L4}	[0.95 – 1.05] pF	
C_{L5}		
C_{L6}		
C_{L10}		
C_{L11}		
C_{L12}		
PL_1 (PMOS length at N_1)	[0.095 – 0.105] µm	
PW_1 (PMOS width at N_1)	[9.5 – 10.5] µm	
NL_1 (NMOS length at N_1)	[0.095 – 0.105] µm	
NW_1 (NMOS width at N_1)	[9.5 – 10.5] µm	
PL_3 (PMOS length at N_4)	[0.095 – 0.105] µm	
PW_3 (PMOS width at N_4)	[9.5 – 10.5] µm	
NL_3 (NMOS length at N_4)	[0.095 – 0.105] µm	
NW_3 (NMOS width at N_4)	[9.5 – 10.5] µm	

The number of SPICE simulations incurred in constructing the forward model is $N_{\text{sim}} = 680$. By leveraging the forward model, the original $P = 60$ mixed dimensions of the example

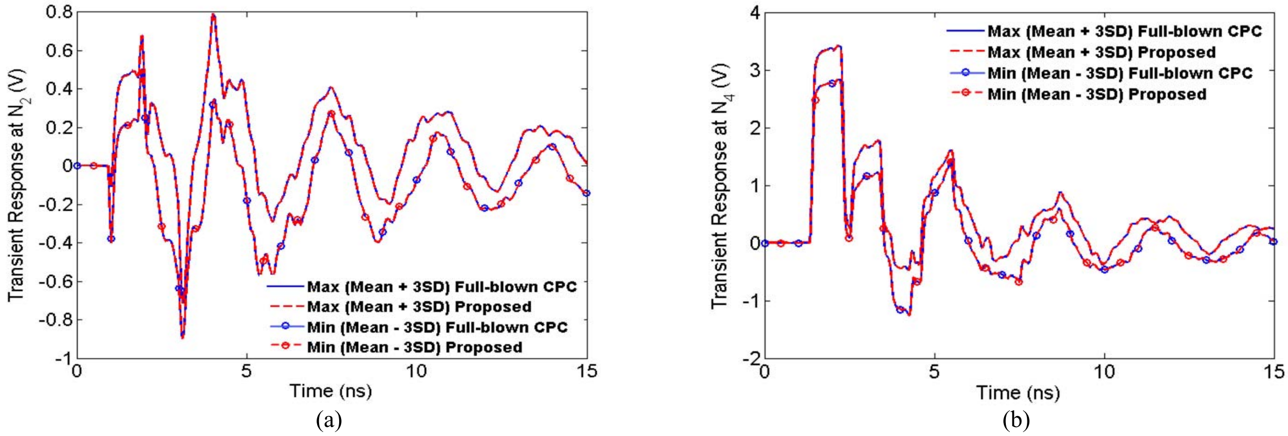


Fig. 8. Statistics of the transient responses at nodes N_2 and N_4 of Fig. 6. (a) Maximum bound of mean plus three times the SD and minimum bound of mean minus three times the SD of the transient response at node N_2. (b) Maximum bound of mean plus three times the SD and minimum bound of mean minus three times the SD of the transient response at node N_4.

TABLE VI
CPU COST ANALYSIS EXAMPLE 3

Approach	Original # Dimensions (P)	Reduced # Dimensions (p)	# SPICE Simulations	CPU cost (in minutes)	Speedup
Full-blown CPC Metamodel	60	-	79,422	622.14	-
Work of [27]		43	30,601	239.71	2.60
Proposed reduced dimensional metamodel (no priority based node selection)		20	4,222	33.07	18.81
Proposed reduced dimensional metamodel (using priority based node selection)		20	3,542	27.75	22.42

is reduced to $s = 20$ dimensions. In particular, all mixed dimensions of parameters w_1, w_2, w_3, s_1, s_2, and s_3, and the aleatory dimensions of the PMOS channel length PL_1 and PL_3 are retained. All other dimensions are discarded. As before, this reduction is substantially greater than the reduction from $P = 60$ to $s = 43$ achieved by the work of [26].

The same statistical bounds of Example 1 are again evaluated using the aforementioned three methods, this time for the transient crosstalk responses at nodes N_2 and N_4 of Fig. 6. The corresponding results are compared in Fig. 8. From Fig. 8, it can be concluded that the reduced dimensional PC metamodel shows good agreement with the full-blown PC metamodel.

The total number of SPICE simulations incurred by the full-blown CPC metamodel, the work of [26], and the proposed reduced dimensional CPC metamodel are reported in Table VI. It is observed from Table VI that using the priority based node selection scheme improves the speedup of the proposed metamodel over the full blown CPC metamodel and the work of [26] from approximately 19 and 7 times, respectively, to 22 and 9 times, respectively. For this example, the full set of $N_{\mathrm{sim}} = 680$ SPICE simulations are reused in the priority based node selection scheme.

V. CONCLUSION

In this paper, a novel reduced dimensional CPC metamodel for the fast mixed epistemic-aleatory uncertainty quantification of high-speed MTL networks is presented. The construction of this metamodel is based upon a new sensitivity sweeping algorithm. This algorithm facilitates extracting global sensitivity information regarding each epistemic and aleatory network dimension that otherwise could not be obtained due to the variability in the response statistics. Based on this global sensitivity information, the most unimportant network dimensions are objectively identified and removed. Consequently, the number of network dimensions are reduced. Numerical examples reveal that constructing a CPC metamodel using only the reduced dimensions is significantly more efficient than constructing the full-blown CPC metamodel without much loss in accuracy. Furthermore, an iterative model enrichment strategy and a priority-based node selection strategy to curb the CPU overheads of this sensitivity sweeping algorithm have also been developed.

REFERENCES

[1] S. De Ridder *et al.*, “A generative modeling framework for statistical link analysis based on sparse data,” *IEEE Trans. Compon., Packag., Manuf. Technol.*, vol. 8, no. 1, pp. 21–30, Jan. 2018.

[2] Y. Ye, D. Spina, P. Manfredi, D. V. Ginste, and T. Dhaene, "A comprehensive and modular stochastic modeling framework for the variability-aware assessment of signal integrity in high-speed links," *IEEE Trans. Electromagn. Compat.*, vol. 60, no. 2, pp. 459–467, Apr. 2018.

[3] A. Biondi, D. V. Ginste, D. De Zutter, P. Manfredi, and F. G. Canavero, "Variability analysis of interconnects terminated by general nonlinear loads," *IEEE Trans. Compon., Packag., Manuf. Technol.*, vol. 2, no. 7, pp. 1244–1251, Jul. 2013.

[4] R. Shen, S. X.-D. Tan, and H. Huo, *Statistical Performance Analysis and Modeling Techniques for Nanometer VLSI Design*. New York, NY, USA: Springer, 2012.

[5] A. Der Kiureghian and O. Ditlevsen, "Aleatory or epistemic? Does it matter?" *Struct. Safety*, vol. 31, no. 2, pp. 105–112, 2009.

[6] F. O. Hoffman and J. S. Hammonds, "Propagation of uncertainty in risk assessments: The need to distinguish between uncertainty due to lack of knowledge and uncertainty due to variability," *Risk Anal.*, vol. 14, no. 5, pp. 707–712, Oct. 1994.

[7] M. S. Eldred, L. P. Swiler, and G. Tang, "Mixed aleatory-epistemic uncertainty quantification with stochastic expansions and optimization-based interval estimation," *Rel. Eng. Syst. Safety*, vol. 96, no. 9, pp. 1092–1113, Sep. 2011.

[8] P. Manfredi, D. V. Ginste, D. De Zutter, and F. Canavero, "Uncertainty assessment of lossy and dispersive lines in SPICE-type environments," *IEEE Trans. Compon., Packag., Manuf. Technol.*, vol. 2, no. 7, pp. 1252–1258, Jul. 2013.

[9] P. Manfredi, D. V. Ginste, D. De Zutter, and F. G. Canavero, "Stochastic modeling of nonlinear circuits via spice-compatible spectral equivalents," *IEEE Trans. Circuits Syst. I, Reg. Papers*, vol. 61, no. 7, pp. 2057–2065, Jul. 2014.

[10] M. R. Rufuie, E. Gad, M. Nakhla, and R. Achar, "Generalized Hermite polynomial chaos for variability analysis of macromodels embedded in nonlinear circuits," *IEEE Trans. Compon., Packag., Manuf. Technol.*, vol. 4, no. 4, pp. 673–684, Apr. 2014.

[11] T.-A. Pham, E. Gad, M. S. Nakhla, and R. Achar, "Decoupled polynomial chaos and its applications to statistical analysis of high-speed interconnects," *IEEE Trans. Compon., Packag., Manuf. Technol.*, vol. 4, no. 10, pp. 1634–1647, Oct. 2014.

[12] Y. Tao, B. Nouri, M. S. Nakhla, M. A. Farhan, and R. Achar, "Variability analysis via parameterized model order reduction and numerical of Laplace transform," *IEEE Trans. Compon., Packag., Manuf. Technol.*, vol. 7, no. 5, pp. 678–686, May 2017.

[13] D. Spina, F. Ferranti, G. Antonini, T. Dhaene, and L. Knockaert, "Efficient variability analysis of electromagentic systems via polynomial-chaos and model order reduction," *IEEE Trans. Compon., Packag., Manuf. Technol.*, vol. 4, no. 6, pp. 1038–1051, Jun. 2014.

[14] D. Spina, F. Ferranti, T. Dhaene, L. Knockaert, G. Antonini, and D. Vande Ginste, "Variability analysis of multiport systems via polynomial-chaos expansion," *IEEE Trans. Microw. Theory Techn.*, vol. 60, no. 8, pp. 2329–2338, Aug. 2012.

[15] A. K. Prasad, M. Ahadi, and S. Roy, "Multidimensional uncertainty quantification of microwave/RF networks using linear regression and optimal design of experiments," *IEEE Trans. Microw. Theory Techn.*, vol. 64, no. 8, pp. 2433–2446, Aug. 2016.

[16] A. K. Prasad and S. Roy, "Accurate reduced dimensional polynomial chaos for efficient uncertainty quantification of microwave/RF networks," *IEEE Trans. Microw. Theory Tech.*, vol. 65, no. 10, pp. 3697–3708, Oct. 2017.

[17] D. Spina, F. Ferranti, G. Antonini, T. Dhaene, and L. Knockaert, "Non-intrusive polynomial chaos-based stochastic macromodeling of multiport systems," in *Proc. IEEE 18th Workshop Signal Power Integrity*, May 2014, pp. 1–4.

[18] A. C. Yücel, H. Bagci, and E. Michielssen, "An ME-PC enhanced HDMR method for efficient statistical analysis of multiconductor transmission line networks," *IEEE Trans. Compon., Packag., Manuf. Technol.*, vol. 5, no. 5, pp. 685–696, May 2015.

[19] P. Manfredi, D. V. Ginste, D. De Zutter, and F. G. Canavero, "Generalized decoupled polynomial chaos for nonlinear circuits with many random parameters," *IEEE Microw. Wireless Compon. Lett.*, vol. 25, no. 8, pp. 505–507, Aug. 2015.

[20] A. Monti, F. Ponci, and F. Valtorta, "Extending polynomial chaos to include interval analysis," *IEEE Trans. Instrum. Meas.*, vol. 59, no. 1, pp. 48–55, Jan. 2010.

[21] R. Schobi and B. Sudret, "Propagation of uncertainties modelled by parametric p-boxes using sparse polynomial chaos expansions," in *Proc. 12th Intl. Conf. Appl. Stats. Probab. Civil Eng.*, Jul. 2015, pp. 1–8.

[22] J. Wu, Z. Luo, N. Zhang, and Y. Zhang, "A new uncertain analysis method and its application in vehicle dynamics," *Mech. Syst. Signal Process.*, vols. 50–51, pp. 659–675, Jan. 2015.

[23] J. Wu, Z. Luo, H. Li, and N. Zhang, "A new hybrid uncertainty optimization method for structures using orthogonal series expansion," *Appl. Math. Model.*, vol. 45, pp. 474–490, May 2017.

[24] Y. Ye, D. Spina, G. Antonini, and T. Dhaene, "Parameterized macromodeling of stochastic linear systems for frequency- and time-domain variability analysis," in *Proc. 22nd Workshop Signal Power Integrity*, May 2018, pp. 1–4.

[25] D. Xiu, *Numerical Methods for Stochastic Computations: A Spectral Method Approach*. Princeton, NJ, USA: Princeton Univ. Press, 2010.

[26] A. K. Prasad and S. Roy, "Mixed aleatory-epistemic uncertainty quantification using reduced dimensional polynomial chaos and parametric ANOVA," in *Proc. IEEE 26th Conf. Electr. Perform. Electron. Packages*, Oct. 2017, pp. 1–3.

[27] Z. Zhang, X. Yang, I. V. Oseledets, G. E. Karniadakis, and L. Daniel, "Enabling high-dimensional hierarchical uncertainty quantification by ANOVA and tensor-train decomposition," *IEEE Trans. Comput.-Aided Design Integr. Circuits Syst.*, vol. 34, no. 1, pp. 63–76, Jan. 2015.

[28] W. Li, G. Lin, and B. Li, "Inverse regression-based uncertainty quantification algorithms for high-dimensional models: Theory and practice," *J. Comput. Phys.*, vol. 321, no. 3, pp. 259–278, Sep. 2016.

[29] R. Tipireddy and R. Ghanem, "Basis adaptation in homogeneous chaos spaces," *J. Comput. Phys.*, vol. 259, pp. 304–317, Feb. 2014.

[30] P. Tsilifis and R. Ghanem, "Reduced Wiener chaos representation of random fields via basis adaptation and projection," *J. Comput. Phys.*, vol. 341, pp. 102–120, Jul. 2017.

[31] J. Jiang, Y. Chen, and A. Narayan, "A goal-oriented reduced basis methods-accelerated generalized polynomial chaos algorithm," *SIAM J. Uncertainty Quantification*, vol. 4, no. 1, pp. 1398–1420, Dec. 2016.

[32] M. Arnst, R. Ghanem, E. Phipps, and J. R. Horse, "Measure transformation and efficient quadrature in reduced-dimensional stochastic modeling of coupled problems," *Int. J. Numer. Methods Eng.*, vol. 92, no. 12, pp. 1044–1080, Dec. 2012.

[33] P. G. Constantine, E. Dow, and Q. Wang, "Active subspace methods in theory and practice: Applications to Kriging surfaces," *SIAM J. Sci. Comput.*, vol. 36, no. 4, pp. A1500–A1524, 2014.

[34] K. Tang, P. M. Congedo, and R. Abgrail, "Adaptive surrogate modeling by ANOVA and sparse polynomial dimensional decomposition for global sensitivity analysis in fluid simulation," *J. Comput. Phys.*, vol. 314, pp. 557–589, Jun. 2016.

[35] G. Blatman and B. Sudret, "An adaptive algorithm to build up sparse polynomial chaos expansions for stochastic finite element analysis," *Probabilistic Eng. Mech.*, vol. 25, pp. 183–197, Apr. 2010.

[36] X. Li, "Finding deterministic solution from underdetermined equation: Large-scale performance variability modeling of analog/RF circuits," *IEEE Trans. Comput.-Aided Design Integr. Circuits Syst.*, vol. 29, no. 11, pp. 1661–1668, Nov. 2010.

[37] J. S. Ochoa and A. Cangellaris, "Random space dimensionality reduction for expedient yield of passive microwave structures," *IEEE Trans. Microw. Theory Techn.*, vol. 61, no. 12, pp. 4313–4321, Dec. 2013.

[38] Z. Zhang, T. Weng, and L. Daniel, "Big-data tensor recovery for high-dimensional uncertainty quantification of process variations," *IEEE Trans. Compon., Packag., Manuf. Technol.*, vol. 7, no. 5, pp. 687–697, May 2017.

[39] H. Rabitz and O. F. Alis, "General formulation of high-dimensional model representations," *J. Math. Chem.*, vol. 50, nos. 2–3, pp. 197–233, 1999.

[40] E. H. Sandoval, F. Anstett-Collin, and M. Basset, "Sensitivity study of dynamic systems using polynomial chaos," *Rel. Eng. Syst. Safety*, vol. 104, pp. 15–26, Aug. 2012.

[41] J. Vlach and K. Singhal, *Computer Methods for Circuit Analysis and Design*. New York, NY, USA: Van Nostrand, 1983.

[42] *HSPICE Reference Manual Version E 2010.12*, Synopsis Inc., Santa Clara, CA, USA, 2010.

Aditi Krishna Prasad (S'14) received the bachelor's degree in electrical engineering from the Dhirubhai Ambani Institute of Information and Communication Technology, Gandhinagar, India, in 2008, the M.S degree from Jawaharlal Nehru Technological University, Hyderabad, India, in 2010, and the Ph.D. degree from Colorado State University, Fort Collins, CO, USA, in 2018.

She is currently an SoC Debug Architect with Intel, Fort Collins. Her current research interests include stochastic algorithms for uncertainty quantification of high-speed systems, global sensitivity analysis using generalized polynomial chaos, and modeling and simulation of high-speed interconnects and power distribution networks.

Sourajeet Roy (S'11–M'13) received the B. Tech. degree in electrical engineering from Sikkim Manipal University, Gangtok, India, in 2006, and the M.E.Sc. and Ph.D. degrees in electrical engineering from Western University, London, ON, Canada, in 2009 and 2013, respectively.

He currently serves as an Assistant Professor with the Department of Electrical and Computer Engineering, Colorado State University, Fort Collins, CO, USA. His current research interests include modeling and simulation of high-speed circuits, signal and power integrity analysis of electronic packages, and uncertainty quantification of microwave/RF circuits.

Dr. Roy is a member of the Technical Program Committee for IEEE Conference on Electrical Performance of Electronic Packaging and Systems and the IEEE Workshop on Signal and Power Integrity. He was a recipient of the Vice-Chancellors Gold Medal at the undergraduate level in 2006, the Queen Elizabeth II Graduate Scholarship in Science and Technology in 2012, and the Ontario Graduate Scholarship in 2012. He currently serves as a reviewer for IEEE TRANSACTIONS ON MICROWAVE THEORY AND TECHNIQUES, IEEE TRANSACTIONS ON COMPONENTS, PACKAGING, AND MANUFACTURING TECHNOLOGY, IEEE TRANSACTIONS ON ELECTROMAGNETIC COMPATIBILITY, and IEEE TRANSACTIONS ON CIRCUITS AND SYSTEMS I: REGULAR PAPERS. He also currently serves as Associate Editor for the IEEE TRANSACTIONS ON COMPONENTS, PACKAGING, AND MANUFACTURING TECHNOLOGY. His student won the Best Poster Paper Award at the 23rd IEEE Conference on Electrical Performance of Electronic Packaging and Systems (EPEPS) in 2014.